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PROGRESS REPORT.

Mr. R. Krishnaswami Aiyar, B.A.—Sub-Assistant Inspector of Schools, Anantapur Range, Anantapur—has been elected a member at the concessional rate.

2. In accordance with Art. VIII (*c*) of the constitution of the Society, the constituted Committee for the ensuing two years has been nominated as under—

1. Mr. E. W. Middlesmast, M.A.—(President);

2. The Hon'ble Mr. R. P. Paranjpye, M.A., B.Sc.,—(Librarian);

3. The Rev. C. P. Pollard—(Treasurer, but Mr. S. Narayana Aiyar will continue to do his duties during his furlough);

4. Mr. R. N. Apte, M.A., L.L.B., F.R.A.S.

5. Mr. Balakram, M.A., I.C.S.

* 6. Rao Bahadur K. Balaram Iyer, B.A.

7. Mr. Gopal Singh Chowla, M.A.

8. Mr. D. D. Kapadia, M.A., B.Sc.

† 9. Mr. M. P. Khareghat, I.C.S. (Retd.)

10. Mr. M. T. Naraniengar, M.A.

11. Mr. S. Narayana Aiyar, M.A., F.S.S.

12. Mr. P. V. Seshu Aiyar, B.A., L.T.

13. Mr. A. C. L. Wilkinson, M.A., F.R.A.S.

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* Mr. R. B. Balaram Iyer will continue till Rev. Pollard returns.

† Mr. Khareghat retires from the committee when Mr. Ramaswami Aiyar accepts his seat.

POONA,
30th November 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theory of Irrational Numbers, Part I.

By Phillip E. B. Jourdain.

(*Continued from page* 175.)

V.

Bolzano was concerned to prove the theorem in the theory of equations that, between two values of the unknown which give the function considered different signs, one real root at least of the equation must lie. He gave a most acute criticism of the previous proofs of this theorem. That proof was to be rejected which some people have carried out by using a conception of the "continuity" of a function, with an admixture of the conceptions of time and motion. "If the two functions," they said "vary according to the law of continuity and, for $x=a$, $f(a)<\phi(a)$, but for $x=b$, $f(b)>\phi(b)$ there must be a value u between a and b for which $f(u)=\phi(u)$." The picture which is at once called up of two bodies in motion cannot be regarded as more than a mere example which does not prove the theorem itself, but must be proved by it According to Bolzano an "incorrect conception of continuity" is here used. "According to a correct explanation, we understand by the phrase that a function $f(x)$ varies for all values of x which lie within or without certain limits according to the law of continuity, only so much that, if x is any such value, the difference $f(x+\omega)-f(x)$ can be made less than any given magnitude, if we can take ω as small as we wish." The supposition that a continuous function can never reach a higher value without passing through all lower ones is a theorem which follows, indeed, from the above conception, but can only be proved by the theorem proved in Bolzano's memoir.

The theorem to be proved evidently rests on the more general theorem that, if two continuous functions $f(x)$ and $\phi(x)$ are such that, for $x=a$, $f(a)<\phi(a)$, and, for $x=b$, $f(b)>\phi(b)$, there must be a value of x, lying between a and b for which $f(x)=\phi(x)$. If $f(a)<\phi(a)$, then, by the law of continuity, we also have $f(a+i)<\phi(a+i)$, if we take i small enough. Thus the property of making f smaller than ϕ is possessed by all values of i which are less than a given one. However, this property is not possessed by all values of i, for $f(b)>\phi(b)$. Now, we have the theorem that, when certain property M is possessed by all values of a variable i which are less than a given one and not by all, there is a greatest value u such that all values of i which are less than u possess the property M. For this value we must have $f(a+u)=\phi(a+u)$. The theorem just mentioned was proved "by showing that those values of i

of which it can be asserted that all lesser ones possess the property M, and those of which this cannot be asserted can be brought as near to one another as we wish. Hence results, for everyone who has a correct conception of magnitude, that the thought of an i which is the greatest of those of which it may be said that all less than it is possessed of the property M, is the thought of an *actual* magnitude".[1]

Bolzano then began his formal demonstration by pointing out that there are infinite series— a geometrical progression whose base is a proper fraction— such that, if only n is chosen large enough, what may be denoted by $s_{n+m}(x)$ differs, however great m may be, from $s_n(x)$ by a magnitude less than any given one however small.[2] "*A fortiori* this must hold of a series whose terms decrease more rapidly than those of a falling geometrical progression".[3] If, then, this condition—that, if ϵ is given to be as small as we please, then, if n is large enough, the difference between s_n and any later sum s_{n+m} is less than ϵ—is fulfilled for an infinite series, "there is a certain stationary magnitude (*beständge Grösse*), and indeed only one, which the terms of this series of sums approximate ever more closely, and to which they can come as close as we please if only we continue the series far enough."[4]

Bolzano's proof,[5] which is a very essential element in the proof[6] of the main theorem that, if a property M is possessed not by all values of a variable magnitude x but by all which are less than a certain u, there is a magnitude U which is the greatest of those of which it can be asserted that all x's less than it have the property M, is as follows: "The supposition that a magnitude X, to which the terms of the series of sums

[1] *Ibid.*, p. 14.

[2] *Ibid.*, pp. 19-20. We may notice that Bolzano considered series whose terms depend on a variable x. Thus, what we denote, following Cauchy, by s , he denoted by $F^n x$, (*ibid.*, p. 39). That, as soon as we consider the sum of n terms of a series as depending on another variable (x) besides n, we get the possibility that, though $s_n(x)$ may tend to a definite limit for each x in the domain considered, there may be no number which is not surpassed by sums of n terms calculated for certain x's, was not noticed by Bolzano. Gauss (cf. M. Pasch, *Veranderliche und Funktion*, Leipzig and Berlin, 1914, pp. 141-142, where a further development of Gauss's conceptions of upper and lower limit was mentioned), in his inaugural dissertation of 1799, remarked, as we have seen above (§ II), that if X can take the value S and not the value U, it does not follow that between S and U there must be a value T which X can reach but not surpass, for it is possible that there is a limit between S and U to which X can approach as closely as we wish without reaching it.

[3] *Ibid.*, p. 20.

[4] *Ibid.*, p. 21.

[5] *Ibid.*, pp. 21-23.

[6] *Ibid.*, pp. 25-89; cf. esp. pp. 27-28.

approach as closely as we wish as the series is continued certainly contains nothing impossible when we do not presuppose that this magnitude is single and invariable. For, if it is to be a magnitude which may vary, we shall be able to take it at every moment so that it comes very near to the term $s_n(x)$, with which it is compared, or indeed coincides completely with it. But that the supposition of a constant magnitude which has the property of approximation to the terms of our series contains no impossibility, follows from the fact that it is possible on this supposition to determine this magnitude as accurately as we wish. For suppose that we wish to determine X so accurately that the difference between the supposed value and the true value of X does not surpass a given magnitude d which is as small as we wish, we have only to seek a term $F_n(x)$ of the given series such that any following term $F_{n+r}(x)$ differs from it by less than $\pm d$. By the supposition, there must be such a $F(x)$. Now I say that the value of $F_n(x)$ differs from the true value of the magnitude X by at most $\pm d$. For if, with any n, r is increased at wish, the difference $X - F_{r+1}(x) = \pm \omega$ can become as small as we wish. But the difference $F_n(x) - F_{n+r}(x)$ remains less than $\pm d$, however great we take r. Thus the difference

$$X - F_n(x) = X - F_{n+r}(x) - [F_n(x) - F_{n+r}(x)]$$

must remain less than $\pm(d+\omega)$. But since this is a constant magnitude with the same n, and ω can be made as small as we wish by increasing r, $X - F_n(x)$ must be equal to or less than $\pm d$. For, if it were greater and, for example, equal to $\pm(d+e)$, then the relation $d+e < d+\omega$, that is to say $e < \omega$, could not possibly subsist if ω is diminished more and more. The true value of X therefore differs from the value of the term $F_n(x)$ by d, and consequently, since we can take d as small as we wish we can determine the true value as accurately as we wish. There is therefore a real magnitude, to which the terms of the series referred to approximate as nearly as we wish if we continue it far enough. But there is only one such magnitude. For if we supposed that besides X there were another constant magnitude Y to which the terms of the series approximate as much as we wish if the series is continued far enough, then the differences $X - F_{n+r}(x)$, and $Y - F_{n+r}(x)$ must be capable of becoming as small as we wish if r is allowed to become great enough. Therefore the same must hold of their own difference $X - Y$; and this is impossible, if X and Y are to be constant magnitudes, unless we suppose that $X = Y$.

VI.

Thus we see that Bolzano did not presuppose the existence of a limit. In fact, his procedure was practically the same as that of Meray

who defined a sequence to be "convergent" if the criterion above referred to is satisfied. Later on, we will consider Meray's theory in detail; but at present we must point out that there is no logical evidence that, in general, such a "convergent" series has a limit at all. Suppose that the set of rational numbers is defined. It is not difficult to give an example of a series of rational terms which is "convergent" and yet has no rational limit. Perhaps nearly everybody would admit, on the ground of unanalyzed ideas of a geometrical "continuum," that, if we represent the various rational sums s_n, s_{n+1}, s_{n+2}, ...of the series in question by points at the corresponding distances on a straight line, these points would define a limiting point. But we can only affirm that the corresponding series has a numerical limit if we assume that to every length corresponds a number, which consequently is not always rational. But this is to assume the existence of irrational numbers.

Proofs of the existence of a limit in the case that what may be called "the Bolzano-Cauchy criterion of convergence" is satisfied were attempted by Hankel[7] and Stolz[8]. If $f(x)$ approaches a limit as x tends to a, we can easily show that, for any possible σ, a possible ϵ exists such that, if $0<\delta<\epsilon$, then

$$[f(a+\epsilon)-f(a+\delta)]<\sigma.$$

The converse proposition that this condition suffices to prove the existence of a limit, must, in Hankel's[9] view, be proved and this can only happen by investigation of the conditions under which there is no limit as x approaches a. In this case we can never choose σ so small that, if $0<\delta<\epsilon$, then

$$[f(a+\epsilon)-f(a+\delta)]<\sigma.$$

Thus, if $f(x)$ does not approach any limit, our supposition is contradicted. Consequently the inverse holds.

Stolz's[10] proof by means of the "limits of indetermination" of $F_n(x)$ introduced by P. du Bois-Reymond, is only apparent, since it presupposes the existence of a limit of indetermination just as Bolzano's theorem pre-supposes the definition of a real number.[11]

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[7] Article "Grenze" in Ersch and Gruber's *Allg Encyclopadie*, 1. Sekt., Vol. XC. 1871, p. 193. Hankel, in § 19 of this article, was the first mathematician to call attention to the merits of Bolzano's mathematical work.

[8] On p. 260 of his article "B. Bolzano's Bedeutung in der Geschichte de Infinitesimalrechnung", *Maths. Ann.*, Vol. XVIII, 1861, pp. 255-279. See *Ostwald's Klassiker*, No. 153, p. 42.

[9] *Ibid.*, p. 107. cf. p. 95.

[10] *Loc. cit.* p. 260.

[11] *Ostwald's Klassiker*, No. 153, p. 42.

VII

Du Bois-Reymond's treatment of the questions lying at the basis of the theory of convergence is very instructive. He noticed[12] that the principles which held for the convergence of integrals were also fundamental for the convergence of all the series known to him, and that besides this, they serve to prove the then unproved convergence of certain of the series used in mathematical physics and which represent arbitrary functions. These principles can be combined into a small group of theorems about the sum

$$u_1v_1+u_2v_2+\ldots\ldots,$$

and correspond to the cases where the sum $u_1+u_2+\ldots$is (1) absolutely convergent, (2) conditionally convergent, (3) indeterminate. Putting $u_1+u_2+\ldots+u_n=U_n$ and $\lim_{n=\infty} U_n=U$, du Bois-Reymond distinguished four cases :

1. U is finite and determined, as in the case of the series $1-\frac{1}{2}+\frac{1}{3}-\ldots$;
2. U is infinite, as in the case of $1+\frac{1}{2}+\frac{1}{3}+\ldots$;
3. U cannot be infinite, but is not determined, as in the case $1-1+1-\ldots$;
4. U can be infinite and is not determined, as in the case $1-2+2-3+3-\ldots$

The object of du Bois-Reymond's memoir of 1870 was to exhibit some new relations between the first and the third case.

Du Bois-Reymond[13] formulated the fundamental principle of convergence as follows : "U is finite and determined or U is not, according as the sum $u_m+u_{m+1}+\ldots+u_n$ has zero for its limit when m and n become infinite with any relative velocity." On this principle he remarked: It is easy to show that if U is finite and determined, $u_m+\ldots+u_n$ must vanish when both m and n become infinite. It is also easy to show that U cannot be finite and determined if $u_m+\ldots+u_n$ does not vanish for every relative velocity with which m and n become infinite. It is not so easy to prove that, if $u_m+\ldots+u_n$ always vanishes when m and n become infinite, U cannot become indefinite or infinite. Although the above principle is often used, I do not know a rigid proof of it, and here I will simply assume it and will give the proof, which does not seem to me capable of being carried through without the conception of the definite integral, elsewhere." Du Bois-Reymond, added a note to say that this proof would be given in a work, shortly to be published, on the conver-

[12] *Cf. Ostwald's Klassiker* No. 185, p. 3.

[13] *Ibid.*, p. 5.

gence and divergence of definite integrals and infinite series. This publication never took place [14].

The idea that the sufficiency of Cauchy's criterion can be proved by means of considerations of definite integrals rests on an error in point of principle.[15] In fact, the very proof of the existence of a definite integral presupposes the existence of a limit to a sequence which satisfies Cauchy's criterion. It is true that, in the present case, the question of principle of the existence of a limit is somewhat obscured by du Bois-Reymond's general consideration of series whose sum is indefinite between finite limits. This conception was mentioned by Cauchy in his *Cours* of 1821.[16] A simple example of a function which tends to no one definite limit when the real variable x tends to a certain point, but for which there are finite upper and lower limits, such that the values of the function never go outside these limits, is given by $\sin(1/x)$ as x tends to zero. Here what du Bois-Reymond called "the limits of indefiniteness" are $+1$ and -1. The conception in question seems, if we except Abel, not to have attracted much attention. It was rediscovered and applied widely by du Bois-Reymond in 1870, and again independently discovered and used by Jacques Hadamard in 1888; and just before that time, the importance of the conception in the theory of functions had been emphasized by Otto Stolz and Moritz Pasch." [17]

Du Bois-Reymond [18] proved the following theorem: "Whatever may be the nature of the series $u_1+u_2+\ldots$, there are always two numerical magnitudes B and A, of which the greater B, for infinite values of n, is not surpassed by the series $U_n=u_1+u_2+\ldots u_n$, but on the other hand, is infinitely often reached, while similarly, for n infinite, the sum U_n does not sink below A, but is infinitely often equal to A. Thus, all the values U of which limit $_{n=\infty} U_n$ is capable are included between the limits A and B, and these are the narrowest limits in which we can conclude the values U. We will call the magnitudes A and B the "limits of indefiniteness" (*Unbestimmtheitsgrenzen*) and the difference $B-A$ the "interval of indefiniteness" of the series $u_1+u_2+\ldots\ldots$

"Proof. Let us first prove the existence of the upper limit of indeterminateness B. For every n there must be a least magnitude B_n which is not surpassed by $U_{n+m}=u_1+u_2+\ldots+u_{n+m}$, when m increases from

[14] *Ibid.*, pp. 98-99.

[15] Cf. the remark of Pringsheim's quoted on *ibid.*, p. 98. Du Bois-Reymond's tendency to this confusion in point of principle is also illustrated by pp. 6, 99 (note 4).

[16] *ibid.*, p. 99.

[17] Cf. *ibid.*, pp. 99-100, 102-103.

[18] *Ibid.*, pp. 7-8.

0 to ∞, since otherwise U_{n+m} would have to increase, either only increasingly or alternately increasingly and decreasingly above every limit, which would be contrary to the supposition that U cannot become infinite. This magnitude B_n does not increase for any value of n, that is to say $B_n - B_{n+1}$ is never negative, because we have supposed B_n is the least value which is not surpassed by U_{n+m}. Since now B_n never increases with increasing n, B_n must either approach a finite definite limit or not. In the latter case, B_n can only approach negative infinity [19], but this is excluded by the supposition. Thus B_n approaches as n increases, a finite and determined limit, which we will call B. This limit must be reached infinitely often for $n=\infty$. For, if it does not reach this limit infinitely often, it would not be reached any more for a finite value N of n and onwards. Then

$$B = B_n = U_n = u_1 + u_2 + \dots + u_n,$$

and B_{N+1} would be smaller than B_n by a finite magnitude. But then B could never be equal to B_n, since B is the limit to which the least values B_n must approach as n increases to infinity. Therefore U is infinitely often equal to B. The second part of the proof concerning A is proved quite similarly so that I will pass it over."

In this proof there lies implicitly a reference to a previous definition of irrational numbers, and we saw, when mentioning Stolz's "proof" of the sufficiency of Cauchy's criterion by the help of these very conceptions. This fundamental error of du Bois-Reymond has been strongly emphasized by Pringsheim.[20]

For many years after 1821, Cauchy had, perhaps, an almost exclusive influence on the development of mathematical thought relating to the fundamental question on convergence, and he shares with nobody the glory of first considering the questions of the existence of a definite integral and of the solutions of differential equations. In these questions, convergent sequences of numbers play an essential part. Indeed, the arithmetical formulation of the conception of the definite integral of a continuous function which Cauchy gave in 1823 has been regarded[21] as the decisive turning point in the development of the idea that it is necessary to prove the existence of a limit of a series which is defined

19 "When a function of a variable, while this variable increases to infinity either nowhere decreases or nowhere increases, the function must either approach a finite and definite limit, or it becomes $+\infty$ in the first case and $-\infty$ in the second. It cannot become indefinite."

20 Cf. *ibid.*, pp. 101-102.

21 By Pringsheim, *Encykl. der math. Wiss.*, 1. A. 3, p. 65. Cf. above.

arithmetically. But we must remember that, for Cauchy, what are called "real numbers" had a geometrical foundation. With him, in fact, a "number" is a result of measurement.[22]

VIII.

It was the avoidance of the logical error of defining an irrational number b as the sum Σa_v of an infinite series—for the definition of the sum Σa_v is first obtained by putting it equal to the necessarily already defined number b—that gave rise to the modern theories of irrational numbers. "I believe", said Cantor,[23] "that this logical error, which was first avoided by Weierstrass, was almost universally passed over in earlier times, and was not remarked because it is one of the rare cases in which actual errors do not give rise to any important errors in calculation. Nevertheless, I am convinced that with this error hang together all the difficulties which have been found in the conceptions of the irrational, and, when this error is avoided, the irrational number is established with the same definiteness and clearness in our minds as the rational number". And again,[24] the number b is not defined as the limit of the terms a_v of what Cantor called, in his theory of irrational numbers, a "fundamental series" (a_v), as this would be a similar logical error, because then the existence of a limit would be presumed. In short, we do not need to obtain an irrational number, in the first place, by a limit-process, but, on the contrary, we become convinced of the practicability and evidence of limiting-processes in general by means of our *possession* of it.

IX.

We will now glance at the introduction of limits and irrational numbers in some text-books.

De Morgan's works are especially interesting to the historian of mathematics as showing both the influence of Cauchy, that of the later methods of the older analysts, and that of De Morgan's own powerful and logical mind. Thus we have seen, in one of the preceding notes, that the conception of convergence is formulated after Cauchy, and we must add that the equation

$$-1=1+2+4+8+\dots$$

is justified because the series "is the result of an attempt to procure an arithmetical result, upon an arithmetical process, to represent a

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[22] Cf. §IV above.

[23] *Math. Ann.*, Vol. XXI, 1883, p. 566.

[24] *Ibid.*, pp. 568-569.

quantity which is not arithmetical."[25] In another paper,[26] I have shown that the conceptions of Cauchy and Euler of the "continuity" of a function, were given side by side. And, in the same work, the existence of limits was founded on the postulate which appeared with Peacock, Hankel, and others in the attempt to make algebra a formal science independent of geometry. De Morgan, we know,[27] was keenly conscious of the fact that the system of points on a line is richer, as Dedekind has expressed it, than the system or rational numbers; and he probably did not contemplate the possibility of a purely arithmetical theory of irrational numbers.

Dealing with the subject of *limits*, De Morgan[28] said: "In the questions which occur in arithmetic and algebra, relating to problems the conditions of which can be satisfied only as nearly as we pleased but not exactly, it is usual to create a solution by hypothesis, and to say that we continually approach to that solution, the more nearly we solve the problem. Thus it is never said that there is no such thing as x, which makes x^2 actually equal to 6; but it is said that there is suck a thing as the square root of 6, and it is denoted by $\sqrt{6}$. But we do not say we actually find this, but that we *approximate* to it. This non-existing limit, if we may so call it, actually has a more definite existence in geometry than in arithmetic, but only when we make a sort of supposition which is practically as impossible as the extraction of the square root of 6 in arithmetic. Let there be such things as geometrical lines, namely, lengths which have no breadth or thickness, and let it be competent to us to mark off points which divide one part of a line from another, without themselves filling any portion of space; then it is shown in Euclid that the side of a square which contains six square units is a line, which, when we come to apply arithmetic to geometry, must be called $\sqrt{6}$ whenever our arbitrary linear unit is called 1. And the lines represented by the preceding twelve fractions will, in such case, be a set of lines which being always greater than the line in question, yet are severally nearer and nearer to it. This line can no more be expressed

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25 *The Differential and Integral Calculus.* London, 1842, p. 224.

26 "The origin of Cauchy's conceptions of a definite integral and of the continuity of a function", *Isis*, Vol. I, 1914, pp. 661-703.

27 Cf. his little book *On the Connexion of Number and Magnitude: an Attempt to Explain the Fifth Book of Euclid*, London, 1836.

28 *Calculus*, pp. 7-8.

by means of an arithmetical fraction than $\sqrt{6}$." A "limit" was defined[29] as a certain quantity 1, so that..." [30].

Arthur Rösler[31] examined sixteen text-books of dates varying from 1776 to 1881 with respect to the definition of irrational numbers, and found that, in essentials, they all came under the four heads:

"1. An irrational number is the ratio of two continuous incommensurable, homogeneous magnitudes.

"2. An irrational number is a number (numerical form, numerical magnitude), whose value can be given only approximately but up to any desired degree of approximation by means of rational numbers.

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[29] *Ibid.*, p. 9.

[30] In this connexion, an interesting study is afforded by G. Chrystal's *Algebra, an Elementary Text-book for Higher Classes of Secondary Schools and for Colleges* Part ii., Edinburgh, 1889. On p. 103, Chrystal concluded that Cauchy's criterion of convergence was a sufficient condition for existence of a limit, because, "since S_n is finite for all values of n, the limit S_n cannot be infinite". It is not true that, if $f(x)$ is finite at each x, the upper limit of $f(x)$ is necessarily finite; nor—and it is this fact that concerns us here—is there the least reason for thinking that a series which has not a rational limit has a limit at all, unless the system of real numbers has been previously defined. This Chrystal did not do until his second edition was published. On p. 78 of the same volume, Chrystal, having proved that the values of $(1-1/x)^{-x}$ and $(1+1/x)^{x}$ cannot pass each other when x is increased, concluded: "Hence, when x is increased without limit, $(1-1/x)^{-x}$ must diminish down to a finite limit A, and $(1+1/x)^{x}$ must increase up to a finite limit B", and $A=B$.

In the second edition of this book, however, Chrystal attained to an exposition of the modern theories of irrational numbers.

Axel Harnack in his well-known text-book, though he professed to give an arithmetical account of irrational numbers, really fell back on geometry. On *An Introduction to the Study of the Elements of the Differential and Integral Calculus*, translated by G. L. Cathcart, London: 1891, p. 9, § 6, he says that a fundamental sequence" "defines a number". On p. 8 (§ 5) after determining two monotone sequences (α) and (β) of rational numbers, such that α^n and β^n are always coming nearer to the value a/b (which has no rational nth root), so that they ultimately differ from it as little as we please, Harnack proceeded; "Hence it appears that the numbers α and β themselves also approach more and more to one definite quantity, which, even though it does not exist among rational numbers, is yet called a numerical value, because it is connected with a rational number by a perfectly determinate arithmetical operation." This conclusion may possibly rest on geometrical grounds: it does not rest on logical grounds.

At the beginning of the book (p. 1), Harnack remarked: "To describe thoroughly the phenomena of motion is to assign every circumstance in numbers of concrete units: so that, if the series of numbers is also to enable us to describe motion, it must contain a continuous series of quantities."

[31] *Die neueren Definitionsformen der irrationalen Zahlen und ihre Bedeutung fur die Schule*, Hannover, 1886, pp. 6-7.

"3. An irrational number is a fraction whose numerator and denominator have infinitely many figures, or, what is the same thing, it is an infinite, non-periodic decimal fraction.

"4. An irrational number is the limit—not representable by means of rational numbers—of an infinite sequence of rational magnitudes, or it is the limit of a continually increasing and yet finite magnitude.

"The first and oldest of these definitions is based upon the contents of the tenth book of Euclid's *Elements.* This definition, though correct in itself, has the defect of resting on geometrical considerations, whereas we must wish that the conceptions of arithmetic be developed solely out of arithmetic. Also we must notice that the conception of the ratio between two incommensurable magnitudes can only be developed clearly and understood when the irrational numbers are already introduced.

"The second definition is logically false; it is too narrow a definition as long as we have not given such a development of the conception "number" that the conception of irrational number is contained in it...... How is a clear-thinking scholar to understand this definition, if he is not shown beforehand that, besides the rational numbers, there are yet other numbers just as definite?

"The uselessness of the third definition can be recognized at the first glance; we cannot define a conception which is, for us, to denote something definite by something indefinite unfinished, in a fluid state, a Being by a Becoming.[32]

"The fourth definition, like the second, presupposes that the conception of limit is already given in such a manner that it comprises the conception of irrational number. In the contrary case we have a mere tautology. This error, of tacitly postulating the conception of limit for certain infinite series before a criterion for the existence and a foundation for the idea of such a series-limit was given, was often passed over."

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[32] "An infinite non-periodic decimal fraction can only, I think, be taken to mean the idea (which naturally cannot be completed) of an infinite set of the rational numbers represented by the various finite decimal fractions written That there is a fixed number which is the least number greater than all these rational numbers cannot be concluded from the fact that, geometrically, it seems likely to some of us that there ought to be a point which would correspond to the limit in question."

X

We will now shortly notice a few discussions of irrational numbers which were published before 1872 the year in which the three principal modern theories became known to the public at large by publication.

In 1847 Guilmin[33] attempted to give arithmetical definitions of certain irrational numbers ; and, in the course of his paper he said : [34]

"The roots approached from below [approximations to $\sqrt{2}$] form an increasing sequence ; however, these numbers have an upper limit, for some of them can surpass any one of the corresponding roots from above."

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[33] "Définitions précise arithmetiques des racines et des logarithmes incommensurables", *Nouv. Ann*, Vol VI, 1847, 313-326,. On p. 395 is an announcement of a *Cours complet d' arithmétique....*, by A. Guilmin, Paris, 1847. On p. 313 of his paper, he refers to p. 110 of the same volume. Here is part of an anonymous (by O. Terquem, see the index, pp. 491, 499) paper "Question d'examen. Theorie des exposants de nature quelconque (v. t. V., p. 704)", on pp. 106-113. The part dealing with real irrational exponentials on pp. 109-110 (errata on p. 504) ;

" *Definition.*—The symbol m being a whole number and p being a number which is not a power of index m, the symbol $m\sqrt{p}$ denotes that there exists an infinite series of *finite* numbers, such that by raising them all to the mth power, we obtain a second series which has p for limit ; *i.e.*, a second series of which no term$=p$, but where the difference from above or from below (*en exces ou en défaut*), between the terms and p can become $<$ any given quantity. In the last series the difference between two consecutive terms can also descend below every given quantity ; but it has not a limit assignable by a *finite* number of figures, in any system of numeration ; while the second series has assignable limit. Take, for example, $\sqrt{2}$, we have for first series 1, 3/2, 7/5, 17/12, 19/41,... the differences between the terms continually diminish ; it has not a limit expressible in figures of a numeration. We denote symbolically this limit by $\sqrt{2}$ *i.e.*, by forming the second series 1, 9/4, 49/25, 289/144, etc., the limit 2.

" *General observation.*—Whenever we perform any one of the six arithmetical operations on irrational expressions, we must always understand, if we are to know what we are saying, that we perform these operations on the rational expressions of the first series of which these irrational expressions represent the symbolic limit ".

[34] *Ibid.*, p. 316.

In Joseph Bertrand's Traité d'arithmétique[25] occurs the passage :

"... We suppose that this definition [of an incommensurable number] consists in indicating what are the commensurable numbers which are smaller or greater than itself."

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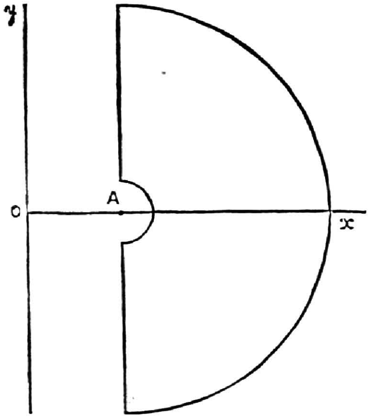
[25] Paris, 1851, p. 231 ; 12th ed., Paris, 1901, p 271. On p. 233 of the 12th ed. Bertrand defined "the magnitude of which $\sqrt{N}$ [N being the square of no whole or fractionary number] is the measure", and said that "it is not possible to define directly an abstract number. If one reflects on the definitions given, given in the simple cases of whole or fractionary numbers, one will see that they are only the indication of the operation by the aid of which the magnitude of which they are the measure is derived from the unit". The same is said for a cube root on p. 255. And on p. 271 : "An incommensurable number can only be defined by indicating how the magnitude which it expresses can be formed by means of the unity. In what follows, we suppose that one can then conceive the magnitude of which it is the measure, as serving as common limit to those which are represented by greater or smaller numbers". "To add or to subtract two incommensurable numbers is to find a number expressing the sum or the difference of the magnitude expressed by the proposed numbers". (*Ibid*).

SHORT NOTES.

On Question 494.

Prove that $\int_0^{\infty} \frac{\phi(a+ix)-\phi(a-ix)}{2ix} dx = \frac{\pi}{2}\left\{\phi(\infty)-\phi(a)\right\}$.

1. The solution given in the I. M. J. of April 1914, p. 65 is not rigorous and the sign of the right hand side is incorrect. Very elementary considerations of the theory of functions afford both a rigorous proof and the conditions under which the theorem is valid.



2. Consider $\int_C \frac{\phi(z)}{z-a} dz$ round a contour consisting of two semicircles centre (a, o) and radii r, R where $z \to o$, $R \to \infty$ and bounded by the broken part of the diameter whose equation is $x=a$.

We have

$$\int_r^R \frac{\phi(a+iy)}{iy} i\,dy - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\phi(a+Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta} d\theta$$

$$-\int_r^R \frac{\phi(a-iy)}{iy} i\,dy + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\phi(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta = 0$$

provided $\phi(z)$ is nowhere infinite within or on the contour of integration.

This becomes

$$\int_0^{\infty} \frac{\phi(a+ix)-\phi(a-ix)}{2ix} dx = \frac{\pi}{2}\left\{\phi(\infty)-\phi(a)\right\}.$$

The conditions to be satisfied by $\phi(z)$ are that $\phi(\infty)$ is finite and that $\phi(z)$ has no infinities to the right of $z=a$.

Example 1. $\phi(z)=e^{-z}$, $a=0$, $\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$.

Example 2. If $\phi(z)$ has infinities within the contour, they must be surrounded by circles of infinitesimal radius; as $\phi(z)=\frac{1}{z-b}$, $b>a$, gives

$$\int_0^\infty \frac{\frac{1}{a-b+ix}-\frac{1}{a-b-ix}}{x}dx+\int_{(b)}\frac{1}{(z-b)(z-a)}dz$$

$$=i\pi\left\{-\frac{1}{a-b}\right\} \text{ since } \phi(\infty)=0.$$

$$\therefore \qquad -2i\int_0^\infty \frac{dx}{(a-b)^2+x^2}+\frac{2i\pi}{b-a}=\frac{i\pi}{b-a},$$

or
$$\int_0^\infty \frac{dx}{(a-b)^2+x^2}=\frac{\pi}{2(b-a)}, \text{ for } b>a.$$

3. Consider $\int \frac{1-e^{-2nz}}{\sinh z}\frac{dz}{z-a}$ taken over a contour bounded by the axis of y, an infinite semicircle, centre the origin, on the positive side of the axis of x, an infinitesimal circle centre (a, o), when a is supposed positive. There are no infinities within the contour; for $\sinh z=0$ gives $e^{2z}=1=e^{2ir\pi}$, but for all these values $\frac{1-e^{-2nz}}{1-e^{-2z}}$ is finite when x is a positive integer.

$$\therefore \int_0^\infty \frac{1-e^{-2nix}}{i\sin x}\frac{1}{ix-a}\,idx-\int_0^\infty \frac{1-e^{2nix}}{-i\sin x}\cdot\frac{1}{-ix-a}(-i)dx$$

$$+2\pi i\frac{1-e^{-2na}}{\sinh a}=0.$$

Simplifying we get

$$\int_0^\infty \frac{a\sin 2nx+x(1-\cos 2nx)}{x^2+a^2}\frac{dx}{\sin x}=\pi\frac{1-e^{-2na}}{\sinh a}.$$

Again consider $\int \frac{1-e^{-2nz}}{\sinh z}\frac{dz}{z+a}$ over the contour bounded by the axis of y and the infinite semicircle, centre the origin, on the positive side of the axis of x, a being positive.

$$\int_0^\infty \frac{x(1-\cos 2nx)-a\sin 2nx}{x^2+a^2}\frac{dx}{\sin x}=0.$$

Hence
$$\int_0^\infty \frac{\sin 2nx}{x^2+a^2}\frac{dx}{\sin x}=\frac{\pi}{a}e^{-na}\frac{\sinh na}{\sinh a}$$

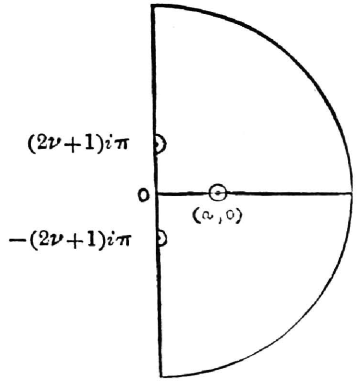
and
$$\int_0^\infty \frac{x\sin^2 nx}{x^2+a^2}\frac{dx}{\sin x}=\frac{\pi}{2}e^{-na}\frac{\sinh na}{\sinh a}.$$

Here n is necessarily a positive integer as $\phi(\infty)$ was assumed to be zero.

4. The special case $2n=-1$ can also be investigated as follows:

Consider $$\int \frac{1}{1+e^{-z}} \frac{dz}{z-a}.$$

The function $1+e^{-z}$ is zero for $z=(2r+1)\pi i$. Surround these points by semicircles as in figure 2 and integrate over the contour thus obtained.



For the point $z=(2r+1)i\pi$, if we take the integral

$$\int_{-r}^{+r} \frac{1}{1-e^{-iu}} \frac{iu}{iu+(2r+1)i\pi-a}=0$$

i. e.

$$\int_{-r}^{+r} \frac{du}{u}=0,$$

we can write

$$\int_0^\infty \frac{1}{1+e^{-ix}} \frac{i\,dx}{ix-a}-\int_0^\infty \frac{1}{1+e^{ix}} \frac{i\,dx}{ix+a}-\pi\phi(\infty)+2\pi i \frac{1}{1+e^{-a}}$$
$$+i\pi\sum_0^\infty -\frac{2a}{(2r+1)^2\pi^2+a^2}=0.$$

For, the integral round the semicircle at $(2r+1)i\pi$ is, by putting $z=(2r+1)i\pi+re^{i\theta}$, equal to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1-e^{-re^{i\theta}}} \frac{re^{i\theta}\,id\theta}{(2r+1)\pi i-a}=\frac{i\pi}{(2r+1)i\pi-a}.$$

Hence for the point $\pm(2r+1)i\pi$, we get $\dfrac{-2i\pi a}{(2r+1)^2\pi^2+a^2}$.

Also $\phi(\infty)=1$.

Hence

$$\int_0^\infty \frac{x\sin\frac{x}{2}-a\cos\frac{x}{2}}{(x^2+a^2)\cos\frac{x}{2}}dx=\pi-\frac{2\pi}{1+e^{-a}}+\sum_0^\infty\frac{2\pi a}{(2r+1)^2\pi^2+a^2}$$

$$=\pi-\frac{2\pi}{1+e^{-a}}+\pi\frac{1-e^{-a}}{1+e^{-a}}.$$

Writing $-a$ for a, and omitting the contour integral round (a, o), we have also

$$\int_0^\infty \frac{x\sin\frac{x}{2}+a\cos\frac{x}{2}}{(x^2+a^2)\cos\frac{x}{2}}dx=\pi-\pi\frac{1-e^{-a}}{1+e^{-a}}.$$

Subtracting we have $\int_0^\infty \frac{dx}{x^2+a^2}=\frac{\pi}{2a}$;

and adding we have $\int_0^\infty \frac{x\tan\frac{1}{2}x}{x^2+a^2}dx=\frac{\pi}{1+e^a}$,

where the principal value of the integral is determined.

5. In the formula of § 2, put $\phi(z)=\frac{e^{-r(z-a)}}{\sinh z}$; thus $\phi(\infty)=0$.

$$\therefore \int_0^\infty\left\{\frac{e^{-rix}}{\sinh(a+ix)}-\frac{e^{rix}}{\sinh(a-ix)}\right\}\frac{dx}{ix}=\pi\left(-\frac{1}{\sinh a}\right)$$

This simplifies to

$$\int_0^\infty \frac{e^a\sin(r+1)x-e^{-a}\sin(r-1)x}{\cosh 2a-\cos 2x}\frac{dx}{x}=\frac{\pi}{2\sinh a}.$$

Put $r=0$, $\int\frac{\sin x}{\cosh 2a-\cos 2x}\frac{dx}{x}=\frac{\pi}{2\sinh a}, \frac{1}{e^a+e^{-a}}.$

and $\int_0^\infty \frac{e^a\sin(2r+1)x}{\cosh 2a-\cos 2x}\frac{dx}{x}=\frac{\pi}{2\sinh a}+e^{-2a}\int_0^\infty\frac{e^a\sin(2r-1)x}{\cosh 2a-\cos 2x}\frac{dx}{x}$

$$=\frac{\pi}{2\sinh a}\left\{1+e^{-2a}+e^{-4a}+-+e^{-2(r-1)a}+e^{-2ra}e\frac{e^a}{e^a+e^{-a}}\right.$$

$$=\frac{\pi}{2\sinh a}\left\{\frac{1-e^{-2ra}}{1-e^{-2a}}+\frac{e^{-2ra}}{1+e^{-2a}}\right\}=\frac{\pi}{2\sinh a}\left\{\frac{1}{1-e^{-2a}}-\frac{2e^{-2(r+1)a}}{1-e^{-4a}}\right\}.$$

$$\therefore \int_0^\infty \frac{\sin(2r+1)x}{\cosh 2a-\cos 2x}\frac{dx}{x}=\frac{\pi}{2\sinh a}\left\{\frac{\pi}{2\sinh a}-\frac{e^{-(2r+1)a}}{\sinh 2a}\right\}.$$

Put $r=1$, and

$$\int_0^\infty \frac{\sin 2x}{\cosh 2a-\cos 2x}\frac{dx}{x}=\frac{\pi e^{-a}}{2\sinh a},$$

and $$\int_0^\infty \frac{e^a \sin 2rx}{\cosh 2a - \cos 2x}\frac{dx}{x} = \frac{\pi}{2\sinh a}\left\{1+e^{-2a}+\ldots+e^{-2(r-1)a}\right\}$$

$\therefore$ $$\int_0^\infty \frac{\sin 2rx}{\cosh 2a-\cos 2x}\frac{dx}{x}=\frac{\pi(1-e^{-2ra})}{4\sinh^2 a}.$$

6. Similarly, applying the same method to $\phi(z)=\frac{e^{-r(z-a)}}{\cosh z}$, we may obtain

$$\int_0^\infty \frac{e^a \sin (r+1)x+e^{-a}\sin (r-1)x}{\cosh 2a+\cos 2x}\frac{dx}{x}=\frac{\pi}{2\cosh a};$$

whence $$\int_0^\infty \frac{\sin (2r+1)x}{\cosh 2a+\cos 2x}\frac{dx}{x}=\frac{\pi}{2\cosh a}\left\{\frac{1}{2\cosh a}+(-)^r\frac{e^{-(2r+1)a}}{\sinh 2a}\right\},$$

and $$\int_0^\infty \frac{\sin 2rx}{\cosh 2a+\cos 2x}\frac{dx}{x}=\frac{\pi(1-(-)^r e^{-2ra})}{4\cosh^2 a}.$$

A. C. L. WILKINSON.

Note on the length of a Circular Arc.

1. Let s denote the arc of a circle of unit radius subtending an angle θ at the centre. Then $s=\theta$. Assume that

$$s=\theta=a_1 \sin p\theta+a_2 \sin p^2\theta+a_3 \sin p^3\theta+\ldots a_n \sin p^n\theta, \qquad \ldots (1)$$

where $p<1$ and $a_1, a_2 \ldots a_n$ are unknown. We shall find values for $a_1, a_2, \ldots a_n$ in terms of p, so that equation (1) may hold good approximately.

Expanding the sines on the right side and neglecting terms beyond θ^{2n-1}, we have the following n equations for $a_1, a_2, \ldots a_n$:

$$\left.\begin{array}{l} a_1p+a_2p^2+\ldots\ldots a_np^n=1 \\ a_1p^3+a_2p^6+\ldots\ldots a_np^{3n}=0, \\ a_1p^5+a_2p^{10}+\ldots\ldots a_np^{5n}=0, \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ a_1p^{2n-1}+a_2p^{4n-2}+\ldots\ldots a_np^{(2n-1)}=0. \end{array}\right\} \qquad \ldots (2)$$

Now, consider the equation

$$\phi(x)\equiv a_nx^{n-1}+a_{n-1}x^{n-2}+\ldots\ldots a_1=0. \qquad \ldots \ldots (3)$$

Its roots are $p^3, p^5, \ldots\ldots p^{2n-1}$, by virtue of (2), and we may write therefore

$$\phi(x)\equiv a_n(x-p^3)(x-p^5)\ldots(x-p^{2n-1})$$
$$\equiv a_n\left\{x^{n-1}-P_1x^{n-2}+P_2x^{n-3}-\ldots\ldots(-1)^{n-1})P_{n-1}\right\} \qquad \ldots (4)$$

Comparing (3) and (4), we deduce that

$$a_n=\frac{a_{n-1}}{-P_1}=\frac{a_{n-2}}{P_2}=\ldots\frac{a_1}{(-1)^{x-1}P_{n-1}}.$$

Also from the first relation in (2)

$$p\phi(p)=1.$$

$$\therefore \quad a_n=\frac{1}{p^n(1-p^2)(1-p^4)\ldots(1-p^{2n-2})} \quad \text{by (4).}$$

Hence

$$a_r=\frac{(-1)^{n-r}\,P_{n-r}}{p^n(1-p^2)\,(1-p^4)\ldots\ldots(1-p^{2n-2})}.$$

2. The error in neglecting powers of θ beyond θ^{2n-1} in the expansion of sin θ is $(-1)^{n+1}\,R_n$, where $R_n<\frac{\theta^{2n+1}}{(2n+1)!}$. Hence a maximum limit to the error in equation (1) is

$$\frac{(-1)^{n+1}\,\theta^{2n+1}}{(2n+1)!}\left\{a_1\,p^{2n+1}+a_2\,p^{4n+2}+\ldots\ldots a_n\,p^{n(2n+1)}\right\}$$

$$=\frac{(-1)^{n+1}\,\theta^{2n+1}}{(2n+1)!}\;\frac{p^{2n+1}\phi(p^{2n+1})}{p\,\phi\,(p)}=\frac{\theta^{2n+1}}{(2n+1)!}p^{n(n+1)}$$

3. *Exnmples* :

(1) Put $p=\frac{1}{2}$, $n=2$; we have $a_1=-\frac{2}{3}$, $a_2=\frac{16}{3}$.

$\therefore \quad s=\theta=\frac{16}{3}\sin\frac{1}{4}\,\theta-\frac{2}{3}\sin\frac{1}{2}\,\theta.$

$=\frac{1}{3}(8H-C)$, [Huyghen's]

where C is the chord of the whole arc, and H is the chord of half the arc.

(2) If $p=\frac{1}{2}$, $n=3$, we find $a_1=\frac{2}{45}$, $a_2=-\frac{16}{9}$, $a_3=\frac{172}{15}$, so that

$s=\theta=\frac{172}{15}\sin\frac{1}{8}\theta-\frac{16}{9}\sin\frac{1}{4}\theta+\frac{2}{45}\sin\frac{1}{2}\theta$

$=\frac{256Q-40H+C}{45}$. [Edward's *Diff. Calc.*, p. 109, Ex. 23(2).]

where Q is the length of quarter the arc.

R. VYTHYNATHASWAMY.

Geometrical Exercises with the Straight Edge.

The straight edge or ruler, as a mathematical instrument, enables as to draw straight lines of indefinite length through one or more points. It is not helpful to set off definite lengths. Geometrical constructions in which the straight edge only is to be used naturally find a place in Projective Geometry, (Geometrie der Lage, Geometrie de Position) in which metrical considerations are not, as in Euclidean Geometry, of primary importance. See : J. W. Russell's *Elementary Treatise on Pure Geometry.*

An attempt is here made to introduce constructions with the straight edge at an earlier stage in connection with Euclid's *Elements.* The fundamental propositions on which the scheme is based are two.

Prop. I. In any triangle, the diagonals of the trapezoid formed by drawing a parallel to one of the sides intersect on the median which bisects that side.

And conversely, if in a triangle, lines be drawn through any point on a median and the extremities of the side to which it is drawn, the line joining the meet of these lines with the other sides is parallel to that side.

Prop. II. If in a triangle OAA′, a line OB be drawn meeting AA′ in B and through any point P on OB, APC, A′PD be drawn meeting OA′ in C and OA in D, then DC meets AA′ in a fixed point B′ which cuts AA′ externally; in other words, B′ is so situated that AB : BA′ : : AB′ : B′A′.

Constructions with the straight edge can be made by paper folding, but the latter also admits of metrical constructions.

Constructions based on Prop. I can be effected with the parallel ruler.

Exercises.

1. Given a finite straight line and its mid-point, draw a line parallel to it through a given point.

2. Given two parallel segments, draw the line which bisects each of them.

3. Three segments and their mid-points being given describe triangles whose sides are parallel to these segments and pass through three given points.

4. Bisect a given straight line, being given,

(1) two lines which are not in the same direction and their mid-points;

(2) a parallelogram;

(3) a circle with its centre;

(4) three equidistant parallel lines.

5. Given two parallel segments produce one of them in increments each equal to its half, and divide the other into any number of equal parts.

6. Construct the join of a given point to the inaccessible point of intersection of two given straight lines.

7. Construct lines which shall be concurrent with a given line and the line joining two given points when this last line cannot be drawn.

8. Through a given point O draw a line cutting the sides BC, CA, AB of a triangle ABC in A′, B′, C′ such that (OA′, B′C′) may be harmonic.

9. Given a parallelogram and a circle find the centre of the circle.

10. Given a circle and its centre.

(1) Inscribe a rectangle in the circle.

(2) Draw the diameters parallel to the sides of the rectangle.

(3) Bisect a chord and its arc.

(4) Bisect an angle whose arms intersect the circle.

(5) Draw from a given point a perpendicular to a given line which cuts the circle.

(6) Draw the diameter which is perpendicular to a given line which does not cut the circle.

(7) Draw through a given point a perpendicular to a given diameter of the circle.

(8) Inscribe

(*a*) an equilateral triangle,

(*b*) a regular pentagon.

11. Given a circle with its centre and an inscribed triangle find the following points:—(*a*) centroid (*b*) in-centre (*c*) points of contact of the in-circle with the sides of the triangle (*d*) orthocentre (*e*) centre of nine-points circle (*f*) point of contact of the in-circle and nine-points circle (*g*) symmedian point (*h*) Brocard points (*i*) Jerabek points.

T. Sundara Row.

The Face of the Sky for January and February 1915.

Sidereal time at 8 p. m.

	December 1914.				*January* 1915.				*February* 1915.		
	H.	M.	S.		H.	M.	S.		H.	M.	S.
1 ...	0	38	24	...	2	40	37	...	4	42	50
8 ...	1	5	59	...	3	8	12	...	5	10	26
15 ...	1	33	35	...	3	35	48	...	5	38	12
22 ...	2	1	11	...	4	3	24	...	6	5	38
29 ...	2	28	47	...	4	31	0				

From this table the constellations visible during the evenings can be ascertained by a reference to their positions as given in the star-charts.

Phases of the Moon.

	January. D.	H.	M.		*February.* D.	H.	M.
Full Moon ...	1	5	51 P.M.				
Last Quarter ...	9	2	43 A.M.	...	7	10	41 A.M.
New Moon ...	15	8	12 P.M.	...	14	10	1 "
First Quarter ...	23	11	2 A.M.	...	22	8	28 "
Full moon ...	31	10	11 "				

Eclipses.

On February 14, there will be an annular solar eclipse, invisible in India.

The Planets.

Mercury is in superior conjunction on January 5 ; and attains its greatest elongation (18° 14′ E) on February 6. It is stationary on February 12. It is in conjunction with the moon on January 16 and on February 15, with Mars on January 2, with Jupiter on February 2 and with Uranus on January 21.

Venus is a morning star. It attains its greatest brilliancy on January 2 and its greatest elongation (46° 54′ W) on February 6. It is in conjunction with the moon on January 12 and February 10. It is in Sagittarius in these months.

Mars is a morning star. It is in Sagittarius in January and near the boundary between Capricornus and Aquarius in February. It is in conjunction with the moon on January 5 and February 13 and with Uranus on February 15.

Jupiter is in conjunction with the sun on February 24. It is in conjunction with the moon on January 18 and February 15. It is near the boundary between Capricornus and Aquarius in January and in Pisces in February.

Saturn is stationary on February 26. It is in conjunction with the Moon on January 27 and February 24. Throughout these months it is in Taurus.

Uranus is in conjunction with the sun on February 1 and with the moon on January 16.

Neptune is in conjunction with the moon on January 3, 30 and February 26.

V. RAMESAM.

SOLUTIONS.

Question 340.

(M. Bhimasena Rao) :—DEF is the pedal triangle of P with respect to a triangle ABC. If AD, BE, CF are concurrent at Q, show that PQ passes through the circumcentre of DEF and through a fixed point

Solution by N. Sankara Aiyar, M. A.

It is easy to see that if P be (f, g, h) then D is $0, g+f\cos C, h+f\cos B$; E is $f+g\cos C, 0, h+g\cos A$; and F is $f+h\cos B, g+h\cos A, 0$.

Hence AD, BE, CF are concurrent if

$$(g+f\cos C)(h+g\cos A)(f+h\cos B)=(g+h\cos A)\times(h+f\cos B)(f+g\cos C) \quad \dots \quad (1)$$

On reduction this becomes

$$\Sigma(f^2g\cos B+fg^2\cos B\cos C)=\Sigma(f^2g\cos A\cos C+fg^2\cos A)$$

i.e., $$\Sigma f(g^2-h^2)(\cos A-\cos B\cos C)=0.$$

This shows that the line PP′, where P′ is the isog. conj. of P, passes through the fixed point $(\cos A-\cos B\cos C)$, etc.

But PP′ also passes through the circumcentre of DEF.

If AD, BE, PP′ are concurrent, it follows that Q lies on PP′ *i.e.*, PQ passes through the fixed point noted above and through the circumcentre of DEF.

Now AD is $\beta(h+f\cos B)-\gamma(g+f\cos C)=0$

BE is $\alpha(h+g\cos A)-\gamma(f+g\cos C)=0$

and PP′ is the line through P and $\cos A-\cos B\cos C$, &c. &c.; *i.e.*, PP′ is

$$\Sigma\alpha(g\,\overline{\cos C-\cos A\cos B}-h\,\overline{\cos B-\cos A\cos C})=0.$$

These three lines are concurrent if

$$(g\,\overline{\cos C-\cos A\cos B}-h\,\overline{\cos B-\cos A\cos C})(h+f\cos B)(f+g\cos C)$$

$$+(h\,\overline{\cos A-\cos B\cos C}-f\,\overline{\cos C-\cos A\cos B})(g+f\cos C)(h+g\cos A)$$

$$+(f\,\overline{\cos B-\cos A\cos C}-g\,\overline{\cos A-\cos B\cos C})(h+f\cos B)(h+g\cos A)=0.$$

Simplifying and writing l, m, n; l', m', n' for $g+h\cos A$, $h+f\cos B$, $f+g\cos C$, $h+g\cos A$, $f+h\cos B$, $g+f\cos C$, we get

$$(l\cos C-l'\cos B)\,mn+(m\cos A-m'\cos C)\,n'l'+(n\cos B-n'\cos A)\,ml'=0;$$

i.e. $lmn\cos C - l'mn\cos B + l'n'm\cos A - l'm'n'\cos C$
$+ l'mn\cos B - l'mn'\cos A = 0;$

i.e. $lmn\cos C - l'm'n'\cos C = 0.,$

which is true, since $lmn = l'm'n'$.

Hence AD, BE, CF are concurrent on PP′.

Question 441.

(S. Ramanujan) :—Shew that

$$(6a^2-4ab+4b^2)^3=(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3$$

and find other quadratic expressions satisfying similar relations.

Solution by S. Narayanan, B.A., L.T.

Let $(la^2-nab+nb^2)^3 \equiv (pa^2+mab-mb^2)^3+(na^2-nab+lb^2)^3 + (ma^2-mab-pb^2)$.

It is proposed to find the most general values of l, m, n, p. The conditions for the identity are seen to be

$$l^3=m^3+n^3+p^3 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and

$$n(l^2-n^2)=m(m^2-p^2). \quad \ldots \quad \ldots \quad (2)$$

Writing (1) and (2) in the forms

$$l^3-n^3=m^3+p^3$$
$$n(l^2-n^2)=m(m^2-p^2),$$

and dividing, we have

$$\frac{l^2+ln+n^2}{n(l+n)}=\frac{m^2-mp+p^2}{m(m-p)};$$

i.e.
$$1+\frac{l^2}{n(l+n)}=1+\frac{p^2}{m(m-p)}.$$

$$\therefore \quad \frac{l^2}{n(l+n)}=\frac{p^2}{m(m-p)},$$

or
$$\frac{n(l+n)}{l^2}=\frac{m(m-p)}{p^2};$$

i.e.
$$\left(\frac{n}{l}+\frac{1}{2}\right)^2=\left(\frac{m}{p}-\frac{1}{2}\right)^2,$$

i e.
$$\frac{n}{l}+\frac{1}{2}=\pm\left(\frac{m}{p}-\frac{1}{2}\right).$$

Hence
$$\frac{n}{l}=\frac{m}{p}-1 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

or
$$\frac{n}{l}=-\frac{m}{p}. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Putting $l=\lambda p$ in (3), we get $n=\lambda(m-p)$.

Substituting in (1) for l and n

$$\lambda^3 p^3 = m^3 + \lambda^3 (m-p)^3 + p^3.$$

$$\therefore \quad \lambda^3 \{ p^3 - (m-p)^3 \} = m^3 + p^3.$$

i.e.
$$\lambda^3 (2p-m)(p^2 - pm + m^2) = m^3 + p^3$$

$$\therefore \quad \lambda^3 (2p-m) = m+p, \text{ since } p^2 - pm + m^2 \neq 0.$$

$$\therefore \quad m(1+\lambda^3) = p(2\lambda^3 - 1).$$

$$\therefore \quad n(1+\lambda^3) = \lambda(m-p) = \lambda(\lambda^3 - 2)p.$$

Hence we may write

$$l = \lambda(\lambda^3+1),\ m = (2\lambda^3 - 1),$$
$$n = \lambda(\lambda^3 - 2),\ p = (\lambda^3+1),$$

which are the general solutions resulting from equation (3).

Again, from (4), we get

$$l = \lambda p,\ n = -m\lambda$$

$$\therefore \quad (p^3\lambda^3 = m^3 - m^3\lambda^3 + p^3, \text{ substituting in (1).}$$

$$\therefore \quad (p^3 + m^3)(\lambda^3 - 1) = 0.$$

Hence, either $p = -m$, and therefore $l = n$; or $\lambda = 1$ and therefore $l = p$, $n = -m$; which are special solutions.

Question 512.

(A. Narasinga Rao):—If $n\,\phi(n) + (n+2)\,\phi(n+2) = \phi(n+1)$, prove that

(i) $\phi(1)\,x - \phi(3)\,x^3 + \phi(5)\,x^5 \ldots = \phi(1)\sin(\tanh^{-1}x)$

(ii) $\phi(0) - \phi(2)\,x^2 + \phi(4)\,x^4 \ldots = \phi(0) + \phi(1)\{\cos\tanh^{-1}x - 1\}$.

Discuss the convergency of the series $\Sigma\phi(n)\,x^n$.

Additional Solution by the Proposer.

Multiply both sides of the given functional ralation by x^{n+1}. Then

$$\frac{1}{x}.\ (n+2)\,f(n+2)\,x^{n+2} + x\ n\,f(n)\,x^n = f(n+1)x^{n+1}$$

$$\frac{1}{x}\,(n+1)\,f(n+1)\,x^{n+1} + x\,(n-1)\,f(n-1)\,x^{n-1} = f(n)\,x^n.$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{x}\ 2\,f(2)\,x^2 + x.0.\,f(0) = f(1).x.$$

By addition

$$\left(x + \frac{1}{x}\right)\{2\,f(2)x^2 + 3\,f(3)x^3 + 4\,f(4)x^4 + \ldots \text{to } \infty\} + x^2 f(1) = \sum_{1}^{\infty} f(r).x^r.$$

Hence if $$u=\sum_{1}^{\infty} f(r)x^r,$$

we have $$(1+x^2)\frac{du}{dx}=u+f(1).$$

$$\therefore \qquad u=Ce^{\tan^{-1}x}-f(1).$$

When $x=0$, $u=0$. $\therefore$ $C=f(1)$.

$$\therefore \qquad \sum_{0}^{\infty} f(r)x^r=f(0)+f(1)\left\{e^{\tan^{-1}x}-1\right\}.$$

On replacing x by ix and equating real and imaginary parts on both sides, we get the two results given.

The series for $\tan^{-1}x$ is convergent or divergent according as

$$(x)\leq \text{ or } >1.$$

Also the series for e^x is convergent for all values of x.

An interesting point about f is its behaviour at infinity. Since $\Sigma f(n)$ is convergent $f(\infty)=0$, for otherwise the series would diverge or oscillate.

Also the test ratio of the series is Lt. $\dfrac{f(n+1)}{f(n)}=\text{Lt.}(t_n) \not> 1.$ (1)

Our functional equation may now be written in the form

$$\frac{n+2}{n}\frac{f(n+2)}{f(n)}+1=\frac{f(n+1)}{f(n)}\frac{1}{n};$$

i. e. $$\underset{n\to\infty}{\text{Lt.}}\ (t_n t_{n+1}+1)=\text{Lt}\left(\frac{tn}{n}\right)=0. \quad \dots \quad \dots \quad (2)$$

If $t_n=t_{n+1}$ then each is equal to $\pm i$ from (2). This is impossible since all terms of the series are real.

Also $t_n \neq 0$ for then t_{n+1} is infinity which cannot be because of (1).

$$\therefore \qquad t_n=-t_{n+1}=1.$$

Hence after a certain term the series consists of two positive and two negative terms.

Question 517.

(T. P. Trivedi, M.A., LL.B.):—Is there any known method of integrating completely

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^2}\cdot\frac{d^4y}{dx^4}+\frac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

which is the well-known differential equation of all conic sections?

Solution by T. P. Bhaskara Shastri and N. Durairajan.

The differential equation should be

$$\left(\frac{d^2y}{dx^2}\right)^2\cdot\frac{d^5y}{dx^5}-5\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\frac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0.$$

The following method is indicated in Examples 49 and 50, Forsyth, p. 497.

Put $\dfrac{d^2y}{dx^2}=z$ and multiply the equation by z^n as an integrating actor : we have

$$z^{n+2}\frac{d^3z}{dx^3}-5z^{n+1}\frac{dz}{dx}\cdot\frac{d^2z}{dx^2}+\frac{40}{9}\cdot z^n\left(\frac{dz}{dx}\right)^3=0.$$

i.e.

$$\frac{d}{dx}\left[z^{n+2}\cdot\frac{d^2z}{dx^2}\right]-\frac{d}{dx}\left[\frac{n+7}{2}\cdot z^{n+1}\cdot\left(\frac{dz}{dx}\right)^2\right]$$
$$+\frac{(n+1)(n+7)}{2}z^n\left(\frac{dz}{dx}\right)^3+\frac{40}{9}\cdot z^n\left(\frac{dz}{dx}\right)^3=0.$$

The equation is integrable if we choose n so as to satisfy the equation

$$\frac{(n+1)(n+7)}{2}+\frac{40}{9}=0,$$

which gives $n=-\dfrac{11}{3}$ or $-\dfrac{13}{3}$.

Taking $n=-\dfrac{11}{3}$, a first integral of the equation is

$$z^{-\frac{5}{3}}\cdot\frac{d^2z}{dx^2}-\frac{5}{3}\cdot z^{-\frac{8}{3}}\cdot\left(\frac{dz}{dx}\right)^2=\text{a constant.}$$

Integrating,
$$z^{-\frac{5}{3}}\frac{dz}{dx}=Ax+B.$$

Integrating again

$$-3\ {}^{-\frac{3}{2}}=A\,x^2+2Bx+2C.$$

Thus we get an equation of the form

$$\frac{d^2y}{dx^2}=(a+2bx+cx^2)^{-\frac{3}{2}}$$

where a, b, c, are arbitrary constants.

Put $G=y(a+2bx+cx^2)^{-\frac{1}{2}}$ or $yX^{-\frac{1}{2}}$

The equation reduces to the form

$$X^2\frac{d^2G}{dx^2}+2X(b+cx)\frac{dG}{dx}=1+G(b^2-ac).$$

$$\therefore\quad \frac{d}{dx}\left[X^2\left(\frac{dG}{dx}\right)^2\right]=[2+2G\,(b^2-ac)]\frac{dG}{dx}.$$

$$\therefore\quad X^2\left(\frac{dG}{dx}\right)^2=d+2G+G^2(b^2-ac).$$

$$\therefore\quad \int\frac{dG}{\{d+2G+G^2(b^2-ac)\}^{\frac{1}{2}}}=e'+\int\frac{dx}{(a+2bx+cx^2)}.$$

Taking the case when b^2-ac is positive, the primitive is

$$\frac{1}{\sqrt{b^2-ac}} \log \left\{ (b^2-ac)G+2+\sqrt{(b^2-ac)(d+2G+G^2\,\overline{b^2-a_c})} \right\}$$

$$=2\frac{1}{\sqrt{b^2-ac}} \log.\, e. \frac{cx+b-\sqrt{b^2-ac}}{cx+b+\sqrt{b^2-ac}};$$

i.e.

$$[(b^2-ac)G+2+\sqrt{(b^2-ac)(d+2G+G^2.\overline{b^2-ac})}]^2$$

$$=e. \frac{cx+b-\sqrt{b^2-ac}}{cx+b+\sqrt{b^2-ac}};$$

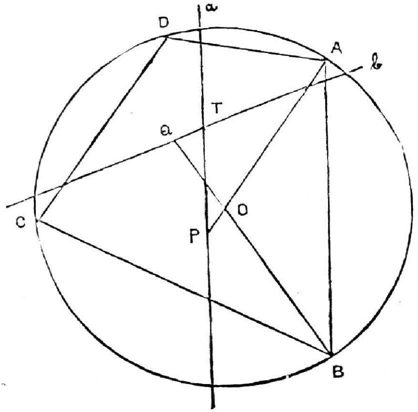
Rationalising and substituting for G, we get the primitive which contains the five arbitrary constants a, b, c, d and e.

Question 520.

(R. Vythyanathaswamy) :—ABCD is a quadrilateral inscribed in a circle, centre O. Denoting the simson line of A *w.r.t* BCD by a and so on, shew that if, a, b, c, d intersect OA, OB, OC, OD in P, Q, R, S, and themselves cointersect at T, then P, Q, R, S, T, O are concyclic.

Remarks by N. Sankara Aiyar, and V. V. S. Narayan.

If, as the question states, P, Q, T, O are cyclic, then



$$P\hat{T}Q = P\hat{O}Q = A\hat{O}B.$$

But $P\hat{T}Q$ is easily to be the supplement of $A\hat{O}B$. Hence the question is incorrect.

Question 542.

(K. V. Anantanarayana Sastri, B. A.) :—Expand $\theta \cot \frac{1}{4}\theta$ in powers of $\cos\theta$.

Solution by K. J. Sanjana and T. P. Trivedi.

In the expansion

$$\frac{\text{vers}^{-1}x}{\sqrt{(2x)}}=1+\frac{1}{3}\,\frac{x}{4}+\frac{1\cdot 3}{5}\,\frac{x^2}{4^2|\underline{2}}+\frac{1\cdot 3\cdot 5}{7}\,\frac{x^3}{4^3|\underline{3}}+\ldots,$$

put $\text{vers}^{-1}x=\theta$; then $x=1-\cos\theta$, $\sqrt{(2x)}=2\sin\frac{\theta}{2}$, and we get

$$\frac{\theta}{2\sin\frac{\theta}{2}}=1+\frac{1}{3}\,\frac{(1-\cos\theta)}{4|\underline{1}}+\frac{1\cdot 3}{5}\,\frac{(1-\cos\theta)^2}{4^2|\underline{2}}+\ldots\ldots$$

Again, in the identity,

$$\theta\cot\theta=1-\frac{\sin^2\theta}{3}-\frac{2}{3}\,\frac{\sin^4\theta}{5}-\frac{2\cdot 4}{3\cdot 5}\,\frac{\sin^6\theta}{7}-\ldots,$$

change θ to $\frac{1}{2}\theta$; as $\sin^2\frac{1}{2}\theta=\frac{1}{2}(1-\cos\theta)$, we get

$$\frac{\theta}{2}\cot\frac{\theta}{2}=1-\frac{1}{3}\,\frac{(1-\cos\theta)}{2}-\frac{2}{3}\,\frac{(1-\cos\theta)^2}{2^2\cdot 5}-\ldots\ldots$$

Adding up the two results, the left side gives $\frac{\theta}{2}\left(\text{cosec}\,\frac{\theta}{2}+\cot\frac{\theta}{2}\right)$

i. e., $\frac{\theta}{2}\cot\frac{\theta}{4}$; the right side gives

$$2-\frac{(1-\cos\theta)}{3\cdot 2}\left\{\frac{1}{1}-\frac{1}{2|\underline{1}}\right\}-\frac{(1-\cos\theta)^2}{5\cdot 2^2}\left\{\frac{2}{3}-\frac{1\cdot 3}{2^2\cdot 2}\right\}$$
$$-\frac{(1-\cos\theta)^3}{7\cdot 2^3}\left\{\frac{2\cdot 4}{3\cdot 5}-\frac{1\cdot 3\cdot 5}{2^3|\underline{3}}\right\}\ldots\ldots$$

Twice this series is therefore the required expansion.

Question 543.

(R. Vythynathaswamy) :—Shew that

$$1^r+2^r+3^r+\ldots n^r=(c_2\Delta+c_3\Delta^2+\ldots c_{r+1}\Delta^r)0^r$$

where $c_p=n+1C_p$.

Solution by R. J. Pocock, B.A., B.Sc., F.R.A.S., and A. Narasinga Rao.

We have the general theorem

$$U+U_1+U_2+\ldots U_n=(1+E+E^2+\ldots E^n)U_0.$$
$$=\frac{E^{n+1}-1}{E-1}U_0=\frac{(1+\Delta)^{n+1}-1}{\Delta}U_0.$$
$$=(n+1)U_0+\frac{n(n+1)}{2\,!}\Delta U_0+\ldots$$
$$=(c_1+c_2\Delta+c_3\Delta^2+\ldots)U_0.$$

Putting $U_0=0^r$, we have the required result remembering that if $n>r$ all terms after $\Delta^r.0^r$ vanish.

Question 544.

(SELECTED) :—Evaluate the definite integrals

$$\int_0^\infty e^{-x^2} \sin ax^2\, dx ; \int_0^\infty e^{-x^2} \cos ax^2\, dx.$$

Solution (1) *by R. J. Pocock B. A., B. Sc., F.R.A.S., T. P. Trivedi, M. A., LL. B., and K. J. Sanjana, M. A.* (2) *by J. M. Bose, M·A., B.Sc.*

(1) Let $P=\int_0^\infty e^{-x^2} \cos ax^2 dx,\quad Q=\int_0^\infty e^{-x^2} \sin ax^2 dx.$

Then

$$P—Q\, i=\int_0^\infty e^{-x^2}(\cos ax^2—i \sin ax^2)dx$$

$$=\int_0^\infty e^{-x^2(1+ai)}dx$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{1+ai}}=\frac{\sqrt{\pi}}{2(1+a^2)^{\frac{1}{2}}}(1—ai)^{\frac{1}{2}}$$

$$=\frac{\sqrt{\pi}}{2(1+a^2)^{\frac{1}{2}}}\left(\cos\frac{\theta}{2}—i \sin\frac{\theta}{2}\right),\text{ where } \tan\theta=a.$$

Hence $P=\frac{1}{2}\left(\frac{\pi}{1+a^2}\right)^{\frac{1}{2}}\cos\frac{\theta}{2}=\frac{1}{2\sqrt{2}}\left\{\frac{\pi}{1+a^2}(1+\sqrt{1+a^2})\right\}^{\frac{1}{2}},$

and $Q=\frac{1}{2}\left(\frac{\pi}{1+a^2}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}=\frac{1}{2\sqrt{2}}\left\{\frac{\pi}{1+a^2}(\sqrt{1+a^2}—1)\right\}^{\frac{1}{2}}.$

(2) We have $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

Consider the integral $\int e^{-z^2}dz$, taken round a contour consisting of the real axis OA, an arc of a circle AB and a line OB inclined to OA at an angle α.

Since the function has no singularities within the contour

$$\int_C=\int_{OA}+\int_{AB}+\int_{BC}=0.$$

If $OA=R$, then at all points of OA z is real ; hence

$$\int_{OA} e^{-z^2}dz=\int_0^R e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\text{ when R is infinite.}$$

At all points on AB, we may put $z=R(\cos\theta+i\sin\theta)$ where R is constant and θ varies from θ to α.

$$\therefore \quad \int_{AB} = i\mathrm{R}\int_0^{a} e^{-\mathrm{R}^2(\cos 2\theta + i\sin 2\theta)} (i\sin\theta + \cos\theta)\,d\theta$$

$$= i\mathrm{R}\int_0^{a} e^{-\mathrm{R}^2\cos 2\theta} \cdot e^{i(\theta - \mathrm{R}^2\sin 2\theta)}\,d\theta.$$

At all points within the range of integration the real and imaginary parts of

$$\left| \mathrm{R}\int e^{-\mathrm{R}^2\cos 2\theta} e^{i(\theta - \mathrm{R}^2\sin 2\theta)}\,d\theta \right| < \left| \mathrm{R}\int e^{-\mathrm{R}^2\cos 2\theta}\,d\theta \right|.$$

Let $a < \frac{\pi}{4}$ and consider

$$\mathrm{R}\int_0^{\frac{\pi}{4}} e^{-\mathrm{R}^2\cos 2\theta}\,d\theta = \mathrm{R}\left[\int_0^{\frac{1}{8}\pi} + \int_{\frac{1}{8}\pi}^{\frac{1}{4}\pi}\right]$$

Within the range $0 \leq \theta \leq \frac{\pi}{8}$, we have $\cos 2\theta \geq \frac{1}{\sqrt{2}}$

$$\mathrm{R}\left| e^{-\mathrm{R}^2\cos 2\theta}\right| \leq \mathrm{R}\left| e^{-\mathrm{R}^2/\sqrt{2}}\right|$$

which shows that the first integral within the bracket tends to the limit zero when R is indefinitely increased.

Similarly with in the range $\frac{\pi}{8} \leq \theta \leq \frac{\pi}{4}$, $\sqrt{2}\sin 2\theta \geq 1$.

$$\therefore \mathrm{R}\sqrt{2}\mathrm{R}\left| e^{-\mathrm{R}^2\cos 2\theta}\sin 2\theta\right| \geq \mathrm{R}\left| e^{-\mathrm{R}^2\cos 2\theta}\right|$$

$$\therefore \left| \frac{\mathrm{R}\sqrt{2}}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} e^{-\mathrm{R}^2\cos 2\theta}\,d(\cos 2\theta) \right| \geq \left| \mathrm{R}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} e^{-\mathrm{R}^2\cos 2\theta}\,d\theta \right|$$

which shows that the second integral is also zero when $\mathrm{R} = \infty$.

Hence
$$\int_{AB} = 0.$$

and
$$\frac{\sqrt{\pi}}{2}\int_{BO} e^{-z^2}\,dz = 0;$$

at all points on BO put $z = \mathrm{R}(\cos a + i\sin a)$ where R varies from ∞ to zero.

$$\therefore \frac{\sqrt{\pi}}{2} - \int_0^{\infty} e^{-\mathrm{R}^2(\cos 2a + \sin 2a)}\,d\mathrm{R}\,(\cos a + i\sin a) = 0.$$

$$\therefore \int_0^{\infty} e^{-\mathrm{R}^2(\cos 2a + i\sin 2a)}\,d\mathrm{R} = \frac{\sqrt{\pi}}{2}(\cos a - i\sin a).$$

Equating real and imaginary parts

$$\int e^{-R^2 \cos 2\alpha} \cos (R^2 \sin 2\alpha)\, dR = \frac{\sqrt{\pi} \cos \alpha}{2},$$

$$\int e^{-R \cos 2\alpha} \sin (R^2 \sin 2\alpha)\, dR = \frac{\sqrt{\pi} \sin \alpha}{2}.$$

Since R is real, put

$$r^2 \sin 2\alpha = ax^2$$
$$r^2 \cos 2\alpha = x^2.$$

Hence $\tan 2\alpha = a$, so that $\alpha < \frac{\pi}{4}$ so long as a is finite.

Thus
$$\int_0^\infty e^{-x^2} \cos ax^2 dx = \frac{\sqrt{\pi}}{2} \frac{\cos \alpha}{(1+a^2)^{\frac{1}{4}}}$$

$$\int_0 e^{-x^2} \sin ax^2 dx = \frac{\sqrt{\pi}}{2} \frac{\sin \alpha}{(1+a^2)^{\frac{1}{4}}}.$$

Question 547.

(V. K. ARAVAMUDAN)) :—Prove that the oblique trajectories of the system of curves represented by the elimination of t between

$$x = f(t, a)\ ;\ y = \phi(t, a)$$

a being the parameter of the family, are given by replacing a by the complete integral of

$$n\left\{\left(\frac{\partial f}{dt}\right)^2 + \left(\frac{\partial \phi}{dt}\right)^2\right\} + \frac{\partial a}{\partial t}\left\{n\left[\frac{\partial \phi}{dt}\cdot\frac{\partial \phi}{\partial a} + \frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial a}\right] - \frac{\partial \phi}{\partial t}\cdot\frac{\partial f}{\partial a} + \frac{\partial f}{\partial t}\cdot\frac{\partial \phi}{\partial a}\right\} = 0,$$

and hence find the trajectories of a system of confocal ellipses, n being equal to the tangent of the angle of intersection.

Solution by T. P. Bhaskara Shastri and N. Durairajan.

Let the equation of the trajectory be

$$\xi = f(t, a).\ \eta = \phi(t, a) \quad \ldots \quad \ldots \quad \ldots \ (1)$$

where a is to be considered an unknown function of ξ and η to be determined so that the curve may be the trajectory. We now have

$$\frac{dy}{dx} = \frac{\frac{\partial \phi}{\partial t}}{\frac{\partial f}{\partial t}};\ \frac{d\eta}{d\xi} = \frac{\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial a}\cdot\frac{\partial a}{\partial t}}{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}} \quad \ldots \quad \ldots \ (2)$$

and n $$=\frac{\dfrac{dy}{dn}-\dfrac{d\eta}{d\xi}}{1+\dfrac{dy}{dx}\cdot\dfrac{dy}{d\xi}}=\frac{\dfrac{\partial\phi}{\partial t}\left(\dfrac{\partial\phi}{\partial t}+\dfrac{\partial f}{\partial a}\cdot\dfrac{\partial a}{\partial t}\right)-\dfrac{\partial f}{\partial t}\left(\dfrac{\partial\phi}{\partial t}+\dfrac{\partial\phi}{\partial a}\cdot\dfrac{\partial a}{\partial t}\right)}{\dfrac{\partial j}{\partial}\left(\dfrac{\partial f}{\partial t}+\dfrac{\partial f}{\partial a}\cdot\dfrac{\partial a}{\partial t}\right)+\dfrac{\partial\phi}{\partial t}\left(\dfrac{\partial\phi}{\partial t}+\dfrac{\partial\phi}{\partial a}\cdot\dfrac{\partial a}{\partial t}\right)}.$$

$$\therefore\quad n\left\{\left(\frac{\partial f}{\partial t}\right)^3+\left(\frac{\partial\phi}{\partial t}\right)^2\right\}+n\left\{\frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}+\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial a}\cdot\frac{\partial a}{\partial t}\right\}$$
$$-\frac{\partial a}{\partial t}\left\{\frac{\partial\phi}{\partial t}\cdot\frac{\partial f}{\partial a}-\frac{\partial f}{\partial t}\cdot\frac{\partial\phi}{\partial a}\right\}=0.$$

$$\therefore\quad n\left\{\left(\frac{\partial f}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial t}\right)^2\right\}+\frac{\partial a}{\partial t}\left\{n\left[\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial a}+\frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial a}\right]\right.$$
$$\left.-\frac{\partial\phi}{\partial t}\cdot\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\cdot\frac{\partial\phi}{\partial a}\right\}=0.$$

Replacing a by the complete integral of this in equations(1), we get the equations of the oblique trajectories.

Let the equation of the system of confocal ellipses be

$$x=\sqrt{a^2+\lambda}\cos t\qquad y=\sqrt{b^2+\lambda}\sin t.$$

Then, by the preceding part of the problem, the oblique trajectories are given by replacing λ by the integral of

$$n\{(a^2+\lambda)\sin^2 t+(b^2+\lambda)\cos^2 t\}+\frac{d\lambda}{dt}\left\{\frac{1}{2}\left[-\frac{\sqrt{b^2+\lambda}}{\sqrt{a^2+\lambda}}\cos^2 t\right.\right.$$
$$\left.\left.-\frac{\sqrt{a^2+\lambda}}{\sqrt{b^2+\lambda}}\sin^2 t\right]\right\}=0,$$

i.e. $$n\{a^2\sin^2 t+b^2\cos^2 t+\lambda\}=\tfrac{1}{2}\frac{d\lambda}{dt}\sqrt{\frac{a^2\sin^2 t+b^2\cos^2 t+\lambda}{(a^2+\lambda)(b^2+\lambda)}},$$

$$\therefore\quad 2n\,dt=\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)}}\qquad\therefore\quad 2n(t+A)=\cosh^{-1}\frac{\lambda+\dfrac{a^2+b^2}{2}}{\dfrac{a^2-b^2}{2}}.$$

$$\therefore\quad \lambda=\frac{a^2-b^2}{2}.\cosh_2 n(t+A)-\frac{a^2+b^2}{2}.$$

Replacing this value of λ, we get for the trajectories

$$\xi=\sqrt{\frac{a^2-b^2}{2}}.\sqrt{\cosh 2n(t+A)+1}\cos t.$$

$$\eta=\sqrt{\frac{a^2-b^2}{2}}\sqrt{\cosh 2n(t+A)-1}\sin t.$$

which are easily reduced to the form

$$\xi = c \cos t \cosh n(t+A)$$
$$\eta = c \sin t \sinh n(t+A),$$

where $2c$ is the distance between the foci. **(Forsyth : p. 143.)**

Question 551.

(T. P. TRIVEDI, M.A., LL.B.) :—Prove that

$$\iiint (abcdfghlmn)(xyz)^2\, dx\, dy\, dz$$

taken throughout a tetrahedron bounded by the co-ordinate planes and the plane $x/p+y/q+z/r=1$, is equal to

$$\frac{pqr}{60}\{(ap^2+bq^2+cr^2+fqr+grp+hpq)+5(lp+mq+nr)+10d\}.$$

Solution by R. J. Pocock, B.A., B.Sc., F.R.A.S., and R. Vythynathaswamy.

Each term of the Integral is a Dirichlet Integral.

Thus $\iiint ax^2\, dx\, dy\, dz = \dfrac{ap^3qr\Gamma(3)}{\Gamma(5)}\displaystyle\int_0^1 Z^4 dZ = \dfrac{ap^3qr}{60}.$

$$\iiint 2fyz\, dx\, dy\, dz = \frac{2f\cdot p\cdot q^2\cdot r^2\cdot\Gamma(2)\Gamma(2)}{\Gamma(5)}\int_0^1 Z^4 dZ = \frac{f\cdot p\cdot q^2\cdot r^2}{60}.$$

$$\iiint 2lx\, dx\, dy\, dz = \frac{2\cdot l\cdot p^2\cdot q\cdot r\cdot\Gamma(2)}{\Gamma(4)}\int_0^1 Z^3 dZ = \frac{5lp^2qr}{60}.$$

$$\iiint d\cdot dx\, dy\, dz = d\cdot p\cdot q\cdot r\,\frac{1}{\Gamma(3)}\cdot\int_0^1 Z^2 dZ = \frac{d\cdot p\cdot q\cdot r}{6}.$$

The given Integral is therefore equal to

$$\frac{pqr}{60}\{(ap^2+bq^2+cr^2+fqr+grp+hpq)+5(lp+mq+nr)+10d\}.$$

QUESTIONS FOR SOLUTION.

592. (S. Narayana Aiyar, M.A.) :—Show that a^n is equal to

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & (1+a) \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & (1+a)^2 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & (1+a)^3 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ 1 & {}_{n-3}C_1 & {}_{n-3}C_2 & {}_{n-3}C_3 & 1 & 0 & 0 & (1+a)^{n-3} \\ 1 & {}_{n-2}C_1 & {}_{n-2}C_2 & {}_{n-2}C_3 & n-2 & 1 & 0 & (1+a)^{n-2} \\ 1 & {}_{n-1}C_1 & {}_{n-1}C_2 & {}_{n-1}C_3 & {}_{n-1}C_{n-3} & n-1 & 1 & (1+a)^{n-1} \\ 1 & {}_{n}C_1 & {}_{n}C_2 & {}_{n}C_3 & {}_{n}C_3 & {}_{n}C_2 & n & (1+a)^{n} \end{vmatrix}$$

593. (S. Narayana Aiyar, M.A.) :—Shew that

$$\frac{\dfrac{1}{a}+\dfrac{x}{a+1}+\dfrac{x^2}{1\cdot 2}\cdot\dfrac{1}{a+2}+\dfrac{x^3}{1\cdot 2\cdot 3}\cdot\dfrac{1}{a+3}+\ldots \text{ to } \infty}{\dfrac{1}{a+1}+\dfrac{x}{a+2}+\dfrac{x^2}{1\cdot 2}\dfrac{1}{a+3}+\dfrac{x^3}{1\cdot 2\cdot 3}\cdot\dfrac{1}{a+4}+\ldots \text{to } \infty} \text{ is equal to}$$

$$\frac{\dfrac{1}{a}-\dfrac{x}{a(a+1)}+\dfrac{x^2}{a(a+1)(a+2)}-\dfrac{x^3}{a(a+1)(a+2)(a+3)}+\ldots\text{to } \infty}{\dfrac{1}{a+1}-\dfrac{x}{(a+1)(a+2)}+\dfrac{x^2}{(a+1)(a+2)(a+3)}-\ldots\text{to } \infty}.$$

594. (A. A. Krishnaswami Aiyangar, B.A.) :—Establish the following approximate formula for the length of a circular arc :

$$\frac{32768R-5376Q+168H-C}{2735}$$

where C is the chord of the whole arc
H ... half the arc
Q ... quarter the arc
R ... eighth of the arc.

595. (A. A. KRISHNASWAMI AIYANGAR, B. A.) :—ABC is a triangle. D, E, F are the points of contact of the in-circle with the sides BC, CA, AB. D′, E′, F′ are the middle points of AI, BI, CI where I is the incentre. Show that the circles on DD′, EE′, FF′ as diameters are concurrent at the in-Feurbach point. Also prove that a similar property exists for the other Feurbach points.

596. (N. SANKARA AIYAR, M.A.) :—Show that

$$\frac{1}{2}\cdot\frac{1}{2}-\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}-\ldots\ldots=\log\frac{1+\sqrt{2}}{2},$$

and

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}+\ldots\ldots=\log 2.$$

597. (A. NARASINGA RAO) :—Shew that the parabola is the only curve such that the triangle formed by any three points on the curve is proportional to the triangle formed by the tangents at these points.

598. (A, NARASINGA RAO) :—Arrange 13 points so as to form 18 rows of 3 each, no row containing more than 3.

599. (V. V. S. NARAYAN) :—If two circles cut orthagonally find the locus of a point such that the tangents from it to the two circles form a harmonic pencil.

600. (V. V. S. NARAYAN) :—If PQ is a diameter of the circumcircle of a triangle ABC, shew that the axes of the inscribed parabolas whose foci are P and Q intersect at a point R on the circle. Shew further that the axis of the inscribed parabola whose focus is R is perpendicular to PQ.

601. (S. NARAYANAN, B.A., L.T.) :—If X is the Steiner's envelope of a given triangle and Y, the inconic having its centre at the circumcentre, shew that (1) X and Y touch the sides of the triangle in the same three points ; (2) the other six points of intersection of X with the sides of the triangle lie on a conic Z ; (3) Z is concentric with and homethetic to the polar reciprocal of Y with respect to the polar circle of the triangle.

602. (S. KRISHNASWAMI IYENGAR) :—A variable straight line meets two fixed straight lines intersecting at O in the points P and Q. If OP, OQ subtend equal angles at a fixed point A, show that PQ passes through a fixed point B such that BAO is a right angle.

603. (S. KRISHNASWAMI IYENGAR) :—Shew that

$$1-(x)^2+\left(\frac{x^2}{2!}\right)^2-\left(\frac{x^3}{3!}\right)^2+\ldots\ldots=\left(\frac{2}{\pi}\right)\int_0^{\pi}\cos^2(x\sin\theta)d\theta-1.$$

and find the value of

$$1+x^2+\left(\frac{x^2}{2!}\right)^2+\left(\frac{x^3}{3!}\right)^2+\ldots\ldots$$

604. (R. VYTHYNATHSWAMY) :—A quadrilateral PQRS has its vertices on a conicoid. Shew that the plane through PQ which is the harmonic conjugate of the central plane through PQ with respect to the planes PQR, PQS is parallel to the corresponding plane in the case of RS.

605. (S. RAMANUJAN) :—Shew that, when $x=\infty$,

$$\frac{\underline{|(x+a-b)}\quad\underline{|8x+2b}\quad\underline{|9x+a+b}}{\underline{|3x+a-c}\quad\underline{|3x+a-b+c}\quad\underline{|12x+3b}}=\sqrt{\frac{2}{3}}.$$

606. S. RAMANUJAN) :—Shew that

$$\sum_0^{\infty}\left\{\frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}\right\}=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(2+\sqrt{5})\right\}^2.$$

607. (K. J. SANJANA, M.A.) :—Shew how to solve the differential equations

$$\frac{\partial^2 u}{\partial y\,\partial x}-\frac{\partial^2 u}{\partial y\,\partial z}=u(x^2-z^2)\phi(x,y,z),$$

$$\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 u}{\partial x\,\partial z}=u(y^2-z^2)\psi(x,y,z),$$

$$\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y\,\partial z}+\frac{\partial^2 u}{\partial x\,\partial z}=u\theta(x,y,z),$$

where ϕ, ψ, θ denote given functions of x, y and z.

Solve in particular the case when

$$\phi(x,y,z)=\psi(x,y,z)=3/(x^3+y^3+z^3),$$

and $$\theta(x,y,z)=3(x^3+y^3+z^3+2x^2+2y^2+2z^2)(x^3+y^3+z^3).$$

608. (M. BHIMASENA RAO) :—Shew that

(i) $\left(1+e^{-\pi}\right)\left(1+e^{-3\pi}\right)\left(1+e^{-5\pi}\right)\ldots\ldots=2^{\frac{1}{4}}e^{-\frac{\pi}{24}}$

(ii) $\left(1+e^{-\pi\sqrt{3}}\right)\left(1+e^{-3\pi\sqrt{3}}\right)\left(1+e^{-5\pi\sqrt{3}}\right)\ldots=2^{\frac{1}{3}}e^{-\frac{\pi\sqrt{3}}{24}}.$

(iii) $\left(1+e^{-\pi\sqrt{7}}\right)\left(1+e^{-3\pi\sqrt{7}}\right)\ldots\ldots\ldots=2^{\frac{1}{2}}e^{-\frac{\pi\sqrt{7}}{24}}.$

609. (M. BHIMASENA RAO) :—Shew that

$$\text{(i)}\quad \frac{1}{e^{2\pi}-1}+\frac{2^{4n+1}}{e^{4\pi}-1}+\frac{3^{4n+1}}{e^{6\pi}-1}+\frac{4^{4n+1}}{e^{8\pi}-1}+\ldots\ldots\ldots=\frac{B_{2n+1}}{4(2n+1)}$$

$$\text{(ii)}\quad \frac{1}{e^{\pi}+1}+\frac{3^{4n+1}}{e^{3\pi}+1}+\frac{5^{4n+1}}{e^{5\pi}+1}+\ldots=\frac{B_{3n+1}}{4(2n+1)}(2^{4n+1}-1),$$

where B_1, B_2, B_3,...are the numbers of Bernoulli and n is any positive integer in (i), and zero or any positive integer in (ii).

610. (K. V. ANANTHANARAYANASASTRI, B.A.) :—If T is a point on the directrix of a parabola and t_1, t_2 the lengths of the tangents TP, TQ from it to the parabola, shew that the length of the arc PQ of the parabola is

$$\frac{t_1^3+t_2^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{\frac{3}{2}}}\left\{\sinh^{-1}\frac{t_1}{t_2}+\sinh^{-1}\frac{t_2}{t_1}\right\}.$$

611. (T. P. TRIVEDI, M.A., LL.B.) :—Prove that the quadrature of the ellipsoid may be expressed in the form

$$4\pi ab\left\{1-\frac{e^2+e'^2}{6}-S\right\}$$

where e and e' are the eccentricities of the sections by the planes $x=0$ and $y=0$ and

$$S=\sum\frac{e^{2m}\,e'^{2m'}}{(2m+2m'+1)(2m+2m'-1)}\,\frac{(2m)!\,(2m')!}{(m!\,m'!)^2\,4^{m+m'}}$$

612. (N. GANAPATHI SUBBA AIYAR) :—A series of ellipses have a common focus, pass through a given point and have their major axes of the same length. Find their envelope.

185555